Cons-free programs and complexity classes between Logspace and Ptime

Verification and Program Termination 2020

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Computational complexity of languages

- Fix an alphabet Σ. A language is a subset $X \subseteq \Sigma^\star$.
- A Turing machine $\mathcal{M}$ decides X in case, for all $x \in \Sigma^\star$,
  - if $x \in X$, then there is an accepting computation of $\mathcal{M}$ on input x, and
  - if $x \notin X$, then there is a rejecting computation of $\mathcal{M}$ on input x.
- By imposing various resource bounds on $\mathcal{M}$, we get complexity classes.
- PTIME: class of all languages decidable in polynomial time
- LOGSPACE: class of all languages decidable in logarithmic space

Computational complexity of languages

Other relevant models of computation:

	captures $\textsc{Ptime}$ when	natural resources
circuit families	logspace uniform	size, depth
alternating TMs	logspace work tape	time, treesize
AuxPDAs	logspace work tape	time, stack size

- Variants: circuits can have *bounded*, *unbounded*, or *semi-unbounded* fan-in.
- AuxPDAs can be *deterministic* or *nondeterministic*
- close connection between (log) circuit depth, AuxPDA time, (log) ATM time, and ATM treesize.

Computational complexity of languages

These models naturally interpolate between $\textsc{Logspace}$ and $\textsc{Ptime}$

$$\textsc{Logspace} \subseteq \textsc{LogDCFL}, \textsc{NLogspace} \subseteq \textsc{LogCFL} = \mathrm{SAC}^1 \subseteq \mathrm{AC}^1 \subseteq \mathrm{NC}^2 \subseteq \mathrm{SAC}^2 \cdots \subseteq \mathrm{NC} \subseteq \textsc{Ptime}$$

- $\textsc{LogCFL}/\textsc{LogDCFL}$: polynomial-time nondeterminstic/deterministic AuxPDAs
- $\mathrm{AC}^k/\mathrm{SAC}^k/\mathrm{NC}^k$: depth-$O(\log^k n)$ (unbounded/semi-unbounded/bounded fan-in) circuit families
- NC $(= \mathrm{AC} = \mathrm{SAC})$ polylogarithmic depth circuit families, quasi-polynomial time AuxPDAs
- "Orthogonal" to ramification of $\textsc{Ptime}$ by time-$O(n^c)$ TMs.

Program-based complexity

Consider the following first-order functional programming language:

$$
\begin{array}{lcl}
\texttt{p} \in \text{Program} & \leftarrow & \texttt{def}_1\ \texttt{def}_2\ \ldots\ \texttt{def}_m \\
\texttt{def} \in \text{Definition} & \leftarrow & \texttt{f}\ \texttt{x}_1\texttt{x}_2 \ldots \texttt{x}_n = \texttt{e}^f \\
 & & \\
\texttt{e} \in \text{Expression} & \leftarrow & \texttt{x} \mid \texttt{f}\ \texttt{e}_1 \ldots \texttt{e}_n \mid \texttt{0} \mid \texttt{1} \\
 & \mid & \texttt{if}\ \texttt{e}_0\ \texttt{then}\ \texttt{e}_1\ \texttt{else}\ \texttt{e}_2 \\
 & \mid & \texttt{[]} \mid \texttt{cons}\ \texttt{e}_1\ \texttt{e}_2 \\
 & \mid & \texttt{head e} \mid \texttt{tail e} \mid \texttt{null e} \\
 & & \\
\texttt{x} \in \text{Parameter} & \leftarrow & \text{identifier} \\
\texttt{f} \in \text{FcnName} & \leftarrow & \text{identifier, disjoint from Parameter}
\end{array}
$$

Program-based complexity

- Programs can be augmented with any manner of semantics, which give meaning to the phrase "program $\mathtt{p}$ decides language X."
- All natural semantics come with a roughly equivalent measure of running time.
- Moreover program running time is roughly equivalent to time on a Turing machine.
- For any $X \subseteq \Sigma^\star$, $X \in \textsc{Ptime} \iff X$ is decidable by a program in polynomial time.

Cons-free programs

Definition

A program is cons-free in case it contains no occurrence of `cons`.

- For any $X \subseteq \Sigma^\star$, $X \in \textsc{Ptime} \iff X$ is decidable by a cons-free program.
- Let CF be the class of all cons-free programs. For a class $\mathcal{P}$ of programs, we write $\{\{\mathcal{P}\}\}$ to indicate the class of languages decided by some program in $\mathcal{P}$.

Theorem (J '99)

$\{\{\mathrm{CF}\}\} = \textsc{Ptime}$.

Cons-free programs

Definition

A program is tail recursive if all of its recursive calls are operationally final.

- Non-example:

```
even  x = if (null x) then 1 else not(even(tail x))
```

- Example:

```
even x = f x 1
f x b = if (null x) then b else f (tail x) (not b)
```

Theorem (J '99)

$\{\{\text{CF-TR}\}\} = \text{LOGSPACE}$.

Elementary observations

- Note that the maximum depth of the call stack of a cons-free program is bounded by a polynomial in the length of the input.

$\mathtt{f}(\mathtt{x}_0, y_0)$	$\mathtt{f}(\mathtt{x}_1, y_1)$	...	$\mathtt{f}(\mathtt{x}_\ell, y_\ell)$

- Hence (convergent) cons-free programs run in exponential time $(2^{n^{O(1)}})$
- Convergent cons-free tail recursive programs run in polynomial time $(n^{O(1)})$

Elementary observations

- Hence, separating exponential time from polynomial time for cons-free programs would suffice to separate Ptime from Logspace.
- **Motivating question:** What is the expressive power of time-bounded cons-free computation? In particular,
  - Does cons-free running time correspond to any known resource?
  - Are there robust cons-free time classes? (For example, cons-free polynomial time?)

Our results

- At a first approximation, **cons-free running time corresponds to AuxPDA running time**
- (and therefore, to related resources)
- We can characterize $\mathrm{LogDCFL}$, LogCFL, SAC^k, and NC as cons-free time classes.
- Thus: a correlation between cons-free running time and complexity-theoretic hardness, **unlike** ordinary running time!

Our results

- The real story: we can isolate 6 **dialects** of cons-free programs, depending on whether we allow nondeterminism, and what types of output we allow in the recursive function symbols.
- Original language types: e.g., $\mathtt{f} : (\Sigma^\star)^k \to 2$ or $\mathtt{f} : (\Sigma^\star)^k \to \Sigma^\star$.
- Boolean variant: only allow, e.g., $\mathtt{f} : (\Sigma^\star)^k \to 2$.
- Tuple variant: allow output tuples, e.g., $\mathtt{f} : (\Sigma^\star)^k \to (\Sigma^\star)^\ell$.

Our results

Diagram of inclusions:

$$\begin{array}{ccccc} \mathrm{nCF}_0 & \longrightarrow & \mathrm{nCF}_1 & \longrightarrow & \mathrm{nCF}_2 \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{dCF}_0 & \longrightarrow & \mathrm{dCF}_1 & \longrightarrow & \mathrm{dCF}_2 \end{array}$$

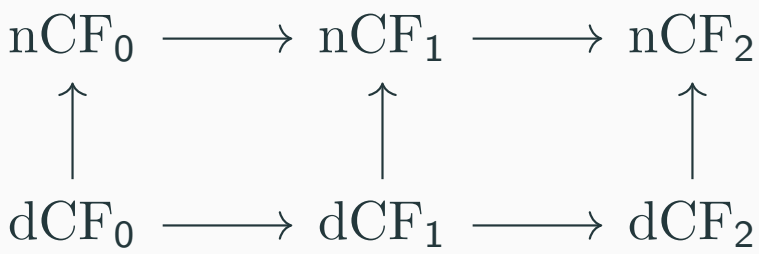
- dCF_1: original cons-free language
- all are equally expressive:

$$\{\{\mathrm{dCF}_0\}\} = \{\{\mathrm{dCF}_1\}\} = \{\{\mathrm{dCF}_2\}\} =$$

$$\{\{\mathrm{nCF}_0\}\} = \{\{\mathrm{nCF}_1\}\} = \{\{\mathrm{nCF}_2\}\} = \textsc{Ptime}$$

Our results

Diagram of inclusions:



- dCF_1: original cons-free language
- However, considering *time-bounded computation* might distinguish them!

Our results

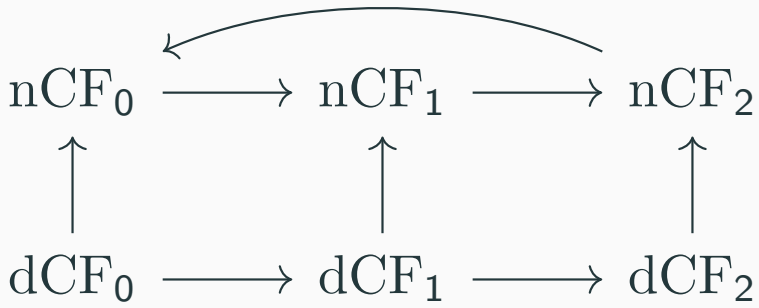
Lemma

nCF_2 programs can be transformed into nCF_0 programs with at most a polynomial loss in running time.

Hence for any class $\mathcal{C}$ of time bounds closed under composition by polynomials,

$$\{\{\mathrm{nCF}_0\text{–}\mathcal{C}\}\} = \{\{\mathrm{nCF}_1\text{–}\mathcal{C}\}\} = \{\{\mathrm{nCF}_2\text{–}\mathcal{C}\}\}.$$

Picture:



Our results

Theorem (with Ben-Amram)

For $k \geq 1$, $\{\{\text{nCF–time}(2^{O(\log^k n)})\}\} = \text{SAC}^k$. Hence,

- $\{\{\text{nCF–poly}\}\} = \text{LogCFL}$. (poly: $n^{O(1)}$)
- $\{\{\text{nCF–quasipoly}\}\} = \text{NC}$. (quasipoly: $2^{(\log n)^{O(1)}}$)

- In fact, SAC^k is known to be closed under complementation.
- **Open problem.** Is this closure realized by a natural program transformation?
- This might also generalize to the closure of NL $(= \{\{\text{nCF–TR}\}\})$ under complementation.

Our results

Theorem (with Ben-Amram)

In the deterministic case,

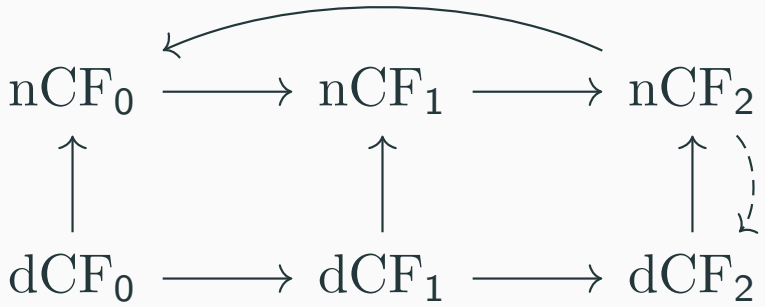
- $\{\{\mathrm{dCF}_2\text{–poly}\}\} = \mathrm{LogDCFL}$.
- $\{\{\mathrm{dCF}_i\text{–quasipoly}\}\} = \mathrm{NC}$ for $i \in \{0, 1, 2\}$.

- Complexity theory tells us that

$$\{\{\mathrm{nCF}\text{–time}(t(n))\}\} \subseteq \{\{\mathrm{dCF}_2\text{–time}(t(n)^{O(\log n)})\}\}.$$

- **Open problem.** Is this containment realized by a natural program transformation?
- $\mathrm{LogDCFL} = \mathrm{LogCFL}$? is the "P vs NP" problem for cons-free programs.

Discussion



Recap of technical open problems. Realize each of the following via natural program translations:

- closure of nCF-classes under complementation
- determinization of cons-free programs, with minimal loss of efficiency
- elimination of tuples/strings in the output, with minimal loss of efficiency

Discussion

Long-term goals:

- *Thesis* (Neil Jones, 1999!): "We maintain that Computability and Complexity theory, and Programming Language and Semantics have much to offer each other, in both directions."
- Interpolate $\textsc{Logspace}$ and $\textsc{Ptime}$ using cons-free programs, and recover all known relationships between complexity classes via natural program transformations.
- Develop a theory of cons-free algorithms, with an eye towards implementing complete problems.